

# Tidal interaction of black holes and Newtonian viscous bodies

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The tidal interaction of a (rotating or nonrotating) black hole with nearby bodies produces changes in its mass, angular momentum, and surface area. Similarly, tidal forces acting on a Newtonian, viscous body do work on the body, change its angular momentum, and part of the transferred gravitational energy is dissipated into heat. The equations that describe the rate of change of the black-hole mass, angular momentum, and surface area as a result of the tidal interaction are compared with the equations that describe how the tidal forces do work, torque, and produce heat in the Newtonian body. The equations are strikingly similar, and unexpectedly, the correspondence between the Newtonian-body and black-hole results is revealed to hold in near-quantitative detail. The correspondence involves the combination  $k_2\tau$  of “Love quantities” that incorporate the details of the body’s internal structure;  $k_2$  is the tidal Love number, and  $\tau$  is the viscosity-produced delay between the action of the tidal forces and the body’s reaction. The combination  $k_2\tau$  is of order  $GM/c^3$  for a black hole of mass  $M$ ; it does not vanish, in spite of the fact that  $k_2$  is known to vanish individually for a nonrotating black hole.

## I. INTRODUCTION AND SUMMARY

### This work and its context

Flanagan and Hinderer have recently shown [1, 2] that the tidal interaction of two neutron stars in the inspiral phase of their orbital evolution leads to measurable effects in the gravitational waves emitted by the binary system. Such a measurement can be used to constrain the radius of each neutron star, and can thus reveal information about the equation of state of nuclear matter. This exciting prospect has provoked a flurry of activity that aims to advance our understanding of tidal interactions between strongly self-gravitating bodies.

Damour and Nagar [3], and independently Binnington and Poisson [4], have formulated a relativistic theory of tidal Love numbers that applies to a neutron star (or any other type of compact object) that is slightly deformed (from a spherical shape) by tidal forces. A tidal Love number is a dimensionless constant of proportionality that appears in the relationship between the tidal forces and the induced body deformation, measured by the multipole moments of the mass distribution. Two types of tidal Love numbers occur in general relativity: An electric-type Love number associated with gravito-electric tidal forces, and a magnetic-type Love number associated with gravito-magnetic tidal forces. In each case the Love number refers to the shared multipole order of the tidal field and the induced deformation; there are two Love numbers for each multipole order.

The electric-type Love number is directly analogous to the tidal Love number of Newtonian gravity (see, for example, Ref. [5]), and indeed, its relativistic definition reduces to the Newtonian definition when the compactness  $C := 2GM/(c^2a)$  of the body is small. (Here  $M$  denotes the body’s mass, and  $a$  is its unperturbed radius; the compactness parameter  $C \leq 1$  measures the importance of relativistic effects on the body’s internal structure.)

The magnetic-type Love number, on the other hand, is a relativistic quantity that vanishes in the Newtonian limit  $C \rightarrow 0$ . The Binnington–Poisson theory [4] applies just as well to black holes, and these authors have shown that the relativistic tidal Love numbers of a nonrotating black hole are all zero.

Other notions of Love numbers have been introduced in Newtonian gravity [5], and these also can be promoted to a relativistic framework. For example, Damour and Nagar [3] have introduced “shape Love numbers” in the relativistic theory of tidally deformed neutron stars; these relate the tidal forces to the displacement of the stellar surface. In a subsequent publication, Damour and Leclercq [6] extended the definition of shape Love numbers to black holes, generalizing a previous analysis by Fang and Lovelace [7]. These works have contributed significantly to our understanding of relativistic tidal interactions.

My purpose in this paper is to contribute even further to this understanding. I show that certain aspects of the tidal interaction of black holes can be interpreted in purely Newtonian terms by exploiting a beautiful analogy between the relativistic physics of black holes and the Newtonian physics of viscous fluids. This analogy was first noticed by Hartle [8, 9], and it was more fully fleshed out in the book *Black Holes: The Membrane Paradigm* [10]. Here the analogy is pushed even further, and unexpectedly, it is revealed to hold in near-quantitative detail. The analogy involves an additional “Love quantity” beyond the tidal and shape Love numbers described previously. The new quantity  $\tau$  is not dimensionless; it possesses the dimension of time, and in Newtonian physics it corresponds to the viscosity-produced delay between the action of the tidal forces and the body’s reaction. I shall refer to  $\tau$  as the “viscous delay.”

### Flux formulae for black holes

The physics to be interpreted concerns the rates at which a black hole changes its mass, angular momentum, and surface area as a result of the tidal interaction with external bodies. These were derived in Ref [11, 12], and the flux formulae are expressed in terms of quantities  $\mathcal{E}_{ab}$  that provide a characterization of the hole's tidal environment. This Cartesian 3-tensor is symmetric and tracefree (STF), and it represents the quadrupole moment of the tidal field acting on the black hole. (Here and below the multipole expansion of the tidal field is truncated to the leading, quadrupole order; generalization to higher multipole orders is possible.)

In Newtonian theory (see, for example, Ref. [13] for a modern treatment), the tidal-moment tensor is defined by first decomposing the Newtonian potential as  $U = U_{\text{body}} + U_{\text{ext}}$ , in which  $U_{\text{body}}$  is associated with the reference body and  $U_{\text{ext}}$  is produced by the external objects. We erect a Cartesian coordinate system  $x^a$  that is attached to the center-of-mass of the reference body, and we define  $\mathcal{E}_{ab}(t) := -\partial_{ab}U_{\text{ext}}$ , in which the spatial derivatives of the external potential are evaluated at  $x^a = 0$ , the position of the center-of-mass. Because  $U_{\text{ext}}$  satisfies Laplace's equation in the body's neighborhood,  $\mathcal{E}_{ab}$  is tracefree in addition to being symmetric in its indices.

We consider a context in which the tidal forces exerted by the external bodies are weak, and in this case it is appropriate to express the external potential as  $U_{\text{ext}} = U_0(t) + g_a(t)x^a - \frac{1}{2}\mathcal{E}_{ab}(t)x^ax^b + \dots$ , a Taylor expansion about the body's center-of-mass. The spatially-constant term  $U_0(t)$  is irrelevant, and  $g_a(t) := \partial_a U_{\text{ext}}$  is responsible for the body's acceleration in the field of the external objects; the remaining terms are responsible for the tidal forces acting on the body.

In general relativity, the tidal moments  $\mathcal{E}_{ab}(v) := C_{0a0b}$  are defined in terms of the components  $C_{0a0b}$  of the space-time Weyl tensor evaluated in a region of spacetime that is far away from the black hole, but still close relative to the external bodies. In this case the indices refer to an inertial frame moving with the black hole, and  $v$  is an advanced-time coordinate on the event horizon. (See Refs. [11, 12] for details.)

For a nonrotating black hole of mass  $M$ , the rates at which the tidal interaction changes the hole's mass and angular momentum are given by Eqs. (8.38) and (8.39) of Ref. [11]; these are

$$\dot{M} = \frac{16}{45}M^6\dot{\mathcal{E}}^{ab}\dot{\mathcal{E}}_{ab} \quad (1.1)$$

and

$$\dot{J} = -\frac{32}{45}M^6(\epsilon^a_{cd}\mathcal{E}^{cb}s^d)\dot{\mathcal{E}}_{ab}, \quad (1.2)$$

respectively. Here an overdot indicates differentiation with respect to advanced-time  $v$ ,  $\epsilon_{abc}$  is the Cartesian permutation symbol, and  $\dot{J} := \dot{J}_as^a$  is the component of the vector  $\dot{J}^a$  in the direction of the (arbitrary) unit

vector  $s^a$ . The rate of change of the surface area is obtained from the first law of black-hole mechanics applied to a nonrotating black hole,  $(\kappa/8\pi)\dot{A} = \dot{M}$ , where  $\kappa = (4M)^{-1}$  is the hole's surface gravity. I employ relativistic units, so that  $G = c = 1$ .

For a rapidly rotating black hole of mass  $M$  and angular-momentum  $J$ , the rate at which the angular momentum changes as a result of the tidal interaction is given by Eq. (9.39) of Ref. [11]; this is

$$\begin{aligned} \dot{J} = & -\frac{16}{45}M^6\Omega_H\left(1 + \sqrt{1 - \chi^2}\right)\left[2(1 + 3\chi^2)(\mathcal{E}_{ab}\mathcal{E}^{ab})\right. \\ & - 3\left(1 + \frac{17}{4}\chi^2\right)(\mathcal{E}_{ab}s^bs^c\mathcal{E}^a_c) \\ & \left. + \frac{15}{4}\chi^2(\mathcal{E}_{ab}s^as^b)^2\right]. \end{aligned} \quad (1.3)$$

Here  $\chi := J/M^2$  is the dimensionless Kerr parameter (which is limited to the interval  $0 \leq \chi < 1$ ), the unit vector  $s^a$  is the direction of the hole's rotation axis, and

$$\Omega_H := \frac{\chi}{2M(1 + \sqrt{1 - \chi^2})} \quad (1.4)$$

is the angular velocity of the event horizon. An equation for  $\dot{M}$  is not available in this case, but at the level of accuracy achieved by Eq. (1.3),  $\dot{A}$  can still be obtained from the first law:  $(\kappa/8\pi)\dot{A} = -\Omega_H\dot{J}$ , where  $\kappa = (2M)^{-1}\sqrt{1 - \chi^2}(1 - \sqrt{1 - \chi^2})^{-1}$  is the surface gravity of a Kerr black hole.

Equations (1.1), (1.2), and (1.3) differ from the original expressions displayed in Ref. [11] by terms involving the gravito-magnetic tidal moments  $\mathcal{B}_{ab}(v)$ . These are switched off because they possess no Newtonian analogues and cannot, therefore, be involved in a correspondence with the tidal dynamics of a Newtonian body. The neglect of the gravito-magnetic terms can be motivated on the basis of a post-Newtonian approximation of the tidal fields [14]; in this context the terms in  $\dot{M}$  and  $\dot{J}$  that involve  $\mathcal{B}_{ab}$  are suppressed relative to those involving  $\mathcal{E}_{ab}$  by factors of order  $(v_{\text{orb}}/c)^2 \ll 1$ , where  $v_{\text{orb}}$  measures the orbital velocity of the external bodies.

### Tidal coupling of a Newtonian, viscous body

The key to a Newtonian interpretation of Eqs. (1.1), (1.2), and (1.3) is the relation (Sec. III D)

$$Q_{jk} = \frac{2}{3}n_2a^5\Omega_H^2C_{jk} - \frac{2}{3}k_2a^5(\mathcal{E}_{jk} - \tau\dot{\mathcal{E}}_{jk}) \quad (1.5)$$

that holds between the mass quadrupole moment  $Q_{jk}$  of a Newtonian body of mass  $M$ , unperturbed radius  $a$ , and angular velocity  $\Omega_H$ , and the quadrupole moment  $\mathcal{E}_{jk}$  of the applied tidal field (an overdot indicates differentiation with respect to time  $t$ ). This relation is formulated in a frame  $x^j$  that rotates uniformly with angular velocity  $\Omega_H$  relative to the global inertial frame (the frame corotates

with the body); this differs from the original frame  $x^a$ , which moves with the body but does not rotate. Equation (1.5) involves the STF tensor  $C_{jk} := \frac{1}{3}\delta_{jk} - s_j s_k$ , where  $s_j$  is a unit vector that points in the direction of the rotation axis. It involves also the dimensionless numbers  $n_2$  (rotational Love number) and  $k_2$  (tidal Love number), and the viscous delay  $\tau$ . The body's mass quadrupole moment is defined by  $Q^{jk} = \int \rho(x^j x^k - \frac{1}{3}r^2 \delta^{jk}) d^3x$ , in which  $\rho$  is the mass density; this STF tensor vanishes when the density profile is spherically symmetric.

The first term on the right-hand side of Eq. (1.5) describes the body's rotational deformation (flattening of the poles, bulging of the equator). The scalings with  $\Omega_H^2$  and  $a^5$  are dictated by dimensional analysis, and the rotational Love number  $n_2$  incorporates the details of the body's internal structure. The second and third terms collectively describe the tidal deformation. Once more the scaling with  $a^5$  is dictated by dimensional analysis, and the tidal Love number  $k_2$  depends on the body's internal structure. The third term is contributed by the body's internal viscosity, and dimensional analysis reveals that  $\tau \propto a\nu/M$ , with  $\nu$  denoting the (averaged) kinematic viscosity of the fluid.

The relation of Eq. (1.5) relies on an assumption that the time scale associated with changes in  $\mathcal{E}_{jk}(t)$  is long compared with the viscous delay  $\tau$ . In this context the second and third terms can be packaged approximately as  $-\frac{2}{3}k_2 a^5 \mathcal{E}_{jk}(t - \tau)$ , which reveals that viscosity does indeed introduce a delay between the action of the tidal forces (described by  $\mathcal{E}_{jk}$ ) and the body's reaction (measured by  $Q_{jk}$ ). The viscous delay produces a misalignment between the figure of the body deformation (described by  $Q_{jk}$ ) and the figure of the tidal forces (measured by  $\mathcal{E}_{jk}$ ). In the case in which the tides are produced by a single external body, the tidal bulge points in a direction that is not quite aligned with the direction of the external body.

Equation (1.5) holds in the body's rotating frame  $x^j$ , and  $\dot{\mathcal{E}}_{jk}$  is the rate of change of the tidal moment as perceived by an observer corotating with the body. In the nonrotating frame  $x^a$  the quadrupole moment is given instead by (Sec. III E)

$$Q_{ab} = \frac{2}{3}n_2 a^5 \Omega_H^2 C_{ab} - \frac{2}{3}k_2 a^5 (\mathcal{E}_{ab} - \tau \dot{\mathcal{E}}_{ab} - \tau \Delta \dot{\mathcal{E}}_{ab}). \quad (1.6)$$

In this expression  $C_{ab} := \frac{1}{3}\delta_{ab} - s_a s_b$ , in which the unit vector  $s_a$  continues to point in the direction of the rotation axis;  $\dot{\mathcal{E}}_{ab}$  is now the rate of change of the tidal moment as perceived by a nonrotating observer, and the last term

$$\Delta \dot{\mathcal{E}}_{ab} := 2\Omega_H \epsilon_{cd(a} \mathcal{E}_{b)}^c s^d \quad (1.7)$$

accounts for the rotation of the body frame relative to the global inertial frame.

## Flux formulae for Newtonian body

The rate at which the tidal forces do work on the Newtonian body is calculated as  $\dot{W} = \frac{1}{2}Q_{ab}\dot{\mathcal{E}}^{ab}$  (Sec. IV A), and substitution of Eq. (1.6) yields

$$\dot{W} = \frac{1}{3}(k_2\tau)a^5 \dot{\mathcal{E}}^{ab}(\dot{\mathcal{E}}_{ab} + \Delta \dot{\mathcal{E}}_{ab}). \quad (1.8)$$

Notice that  $\dot{W}$  is proportional to  $k_2\tau$ : No (net) work is done unless the body is deformed by the tidal forces, and no (net) work is done unless the deformation is delayed with respect to the application of the forces. In the absence of viscosity, the tidal bulge points in the direction of the external body, and the transfer of gravitational energy from the tidal field to the body is fully reversible and does not lead to a net performance of work. In the presence of viscosity, on the other hand, the misalignment of the tidal bulge allows for an irreversible transfer of energy from the tidal field to the body, as described by Eq. (1.8).

The torque exerted on the body by the tidal forces produces a change of angular momentum described by  $\dot{J}_a = -\epsilon_{abc}Q_p^b \mathcal{E}^{pc}$  (Sec. IV B). Substitution of Eq. (1.6) and projection along  $s^a$  yields

$$\Omega_H \dot{J} = -\frac{1}{3}(k_2\tau)a^5 \Delta \dot{\mathcal{E}}^{ab}(\dot{\mathcal{E}}_{ab} + \Delta \dot{\mathcal{E}}_{ab}). \quad (1.9)$$

The result is multiplied by  $\Omega_H$  to convert  $\dot{J}$  into  $\dot{E}_{\text{rot}} = \Omega_H \dot{J}$ , the rate of change of the body's rotational energy. Once more the result is proportional to  $k_2\tau$ : The angular momentum changes if and only if the body is deformed by the tidal forces and the deformation is delayed with respect to the application of the forces. In the absence of viscosity, the tidal bulge points in the direction of the external body, and the tidal forces have no opportunity to exert a torque. In the presence of viscosity, on the other hand, the misalignment of the tidal bulge allows the tidal forces to exert a torque, and the end result is described by Eq. (1.9).

The rate at which viscosity generates heat can be determined from the energy-balance equation (Sec. IV D)  $\dot{Q} = \dot{W} - \Omega_H \dot{J}$ . Substitution of Eqs. (1.8) and (1.9) yields

$$\dot{Q} = \frac{1}{3}(k_2\tau)a^5 (\dot{\mathcal{E}}^{ab} + \Delta \dot{\mathcal{E}}^{ab})(\dot{\mathcal{E}}_{ab} + \Delta \dot{\mathcal{E}}_{ab}). \quad (1.10)$$

## Newtonian interpretation of black-hole fluxes

The black-hole and Newtonian-body results can be put in a close correspondence if we associate the  $\dot{W}$  of Eq. (1.8) with the  $\dot{M}$  of Eq. (1.1), and the  $\dot{J}$  of Eq. (1.9) with the  $\dot{J}$  of Eqs. (1.2) and (1.3). With these associations, the energy-balance equation  $\dot{Q} = \dot{W} - \Omega_H \dot{J}$  becomes the first law of black-hole mechanics,  $(\kappa/8\pi)\dot{A} = \dot{M} - \Omega_H \dot{J}$ , when we also associate  $\dot{Q}$  with  $(\kappa/8\pi)\dot{A}$ ; this

association is natural if we think of  $\kappa/2\pi$  as the black-hole temperature and  $\frac{1}{4}A$  as its entropy.

The nonrotating black hole corresponds to the case in which  $\dot{\mathcal{E}}_{ab}$  dominates over  $\Delta\dot{\mathcal{E}}_{ab}$  in the Newtonian equations; recall from Eq. (1.7) that  $\Delta\dot{\mathcal{E}}_{ab}$  is proportional to  $\Omega_H$  and therefore vanishes when the body is nonrotating. In this case Eqs. (1.8) and (1.9) give

$$\dot{M} = \frac{1}{3}(k_2\tau)a^5\dot{\mathcal{E}}^{ab}\dot{\mathcal{E}}_{ab}, \quad \dot{J} = -\frac{2}{3}(k_2\tau)a^5(\epsilon^a_{cd}\mathcal{E}^{cb}s^d)\dot{\mathcal{E}}_{ab}, \quad (1.11)$$

and these equations bear a striking resemblance with Eqs. (1.1) and (1.2).

The rapidly rotating black hole corresponds to the case in which  $\Delta\dot{\mathcal{E}}_{ab}$  dominates over  $\dot{\mathcal{E}}_{ab}$ ; the changes in the tidal moment are produced almost entirely by the body's own rotation. In this case Eq. (1.9) gives  $\Omega_H\dot{J} = -\frac{2}{3}(k_2\tau)a^5\Delta\dot{\mathcal{E}}^{ab}\Delta\dot{\mathcal{E}}_{ab}$ , and expanding the tensorial expression produces

$$\dot{J} = -\frac{2}{3}(k_2\tau)a^5\Omega_H\left[2(\mathcal{E}_{ab}\mathcal{E}^{ab}) - 3(\mathcal{E}_{ab}s^bs^a\mathcal{E}^a_{cd}\mathcal{E}^{cd})\right]. \quad (1.12)$$

Comparison with Eq. (1.3) reveals another striking similarity, especially if we neglect the terms of order  $\chi^2$  in Eq. (1.3). This is appropriate, because according to Eq. (1.4),  $\chi^2$  can be thought of as a relativistic correction of order  $(v_{\text{rot}}/c)^2$ , in which  $v_{\text{rot}} = a\Omega_H$  stands for the velocity of a fluid element on the surface of the rotating body.

Closer examination reveals that the correspondence produces agreement between *all* numerical coefficients when we make the assignment

$$(k_2\tau)a^5 = \frac{16}{15}\left(\frac{GM}{c^3}\right)\left(\frac{GM}{c^2}\right)^5 \quad (1.13)$$

in the Newtonian equations. (I have restored the factors of  $G$  and  $c$ , which were previously set equal to unity). The equation indicates that as might be expected, the horizon length scale  $GM/c^2$  must play the role of the body radius  $a$ , while the horizon time scale  $GM/c^3$  must play the role of the effective viscous delay  $k_2\tau$ . This last result implies that the horizon can be assigned an effective kinematic viscosity  $k_2\nu \sim GM/c$ .

The picture that emerges from this comparison is one in which the event horizon behaves as if it were a fictitious membrane of viscous fluid; the fluid elements move with the speed of light, and the streamlines are identified with the horizon's generators. This view, of course, is compatible with the *membrane paradigm* of black-hole physics [10].

The correspondence between the black-hole and Newtonian-body fluxes depend on the combination  $k_2\tau$  of “Love quantities,” which is constrained by Eq. (1.13); the constraint applies to rapidly-rotating as well as non-rotating black holes, and it implies that  $k_2\tau$  must be a finite (nonvanishing) quantity. It is known, however, that  $k_2 = 0$  for a nonrotating black hole [4]. One must

therefore think of the combination  $k_2\tau$  as being inseparable for (nonrotating) black holes. While  $k_2$  vanishes individually,  $k_2\tau$  is nevertheless a finite quantity of order  $GM/c^3$ . It is not helpful to think of  $\tau$  as an individuated quantity that happens to be infinite.

## Organization of the paper

In the remaining sections of the paper I elaborate the Newtonian theory of the interaction of a fluid body with a tidal environment characterized by the tidal quadrupole moment  $\mathcal{E}_{ab}(t)$ . In addition to establishing other interesting results, I provide derivations for Eqs. (1.5), (1.6), (1.8), (1.9), and (1.10).

I begin in Sec. II with a warmup exercise in which the fluid's viscosity is set equal to zero; viscosity is introduced next in Sec. III. In Sec. IV I calculate the rates at which the tidal forces do work on the body, change its angular momentum, and lead to the production of heat. For simplicity I perform all calculations assuming that the body consists of an incompressible fluid; general results are obtained from the special cases by inserting Love numbers where appropriate.

## II. TIDAL INTERACTION OF A NEWTONIAN BODY; NONVISCOUS CASE

We consider a body of mass  $M$ , radius  $a$ , and angular velocity  $\Omega_H$  that would be spherical in isolation and in the absence of rotation. The body is made of an incompressible fluid, it rotates rigidly, and it is put in the presence of external objects that exert tidal forces. As a result of its rotation and tidal interaction, the body acquires a deformation. We calculate this deformation, first assuming that the fluid's viscosity can be neglected.

### A. Scales and assumptions

We already introduced the scaling quantities  $M$ ,  $a$ , and  $\Omega_H$ . The tidal interaction is characterized by an external mass scale  $M'$  and a distance scale  $b$ ; it is measured by the tidal moment tensor  $\mathcal{E}_{jk}$ , whose scale is given by  $\mathcal{E} \sim GM'/b^3$ . The time scale associated with changes in the tidal field, as viewed in the global inertial frame, is  $\Omega^{-1}$ , where  $\Omega$  is an angular velocity associated with the motion of the external objects. In the body's rotating frame the time scale becomes  $|\Omega - \Omega_H|^{-1}$ , and the tidal field is static if the angular velocities match at all times. The tidal forces produce a displacement of a fluid element at the body's surface; the length scale of this displacement is  $\xi \sim a^4\mathcal{E}/(GM)$ , and we assume that  $\xi \ll a$ . We also assume that the deformation produced by the rotation is small, but we make no assumption regarding the relative size of these effects.

The velocity scale associated with fluid motions in the rotating frame is  $v \sim \xi|\Omega - \Omega_H|^{-1}$ , or  $v \sim a^4|\Omega - \Omega_H|^{-1}\mathcal{E}/(GM)$ . We assume that  $v$  is much smaller than the linear velocity  $a\Omega_H$  of a fluid element following the body's rotation (as viewed in the global inertial frame). The inequality  $v \ll a\Omega_H$  implies that the Coriolis effect can be ignored, but centrifugal terms will make an appearance in the fluid equations. This assumption constrains the size of the tidal interaction. Assuming that  $\Omega_H$  is larger than or at least comparable to  $\Omega$ , the inequality translates to  $\mathcal{E} \ll GM/a^3$ , or  $(a/b)^3 \ll M/M'$ ; the body must be well separated from the objects that produce the tidal forces. As an additional assumption we take  $v|\Omega - \Omega_H|^{-1}$ , the velocity scale divided by the tidal time scale, to be much smaller than  $\mathcal{E}a$ , the scale associated with the tidal forces (per unit mass) acting within the body. This inequality allows us to neglect the inertial term  $\partial_t v$  in the fluid equation. It translates to  $(\Omega - \Omega_H)^2 \ll GM/a^3$ , and the assumption constrains the size of the angular velocities; they must be small compared with the Keplerian angular velocity of a particle at the body's surface. All the assumptions made here are reasonable, and they do not severely restrict the range of physical situations that can be considered.

### B. Fluid equations; nonrotating frame

We first formulate the fluid equations in a frame  $x^{\bar{a}}$  that moves with the body's center-of-mass but does not rotate relative to the global inertial frame. (This frame was denoted  $x^a$  in Sec. I, and here we put an overbar on the index to increase notational clarity.) The frame  $x^{\bar{a}}$  is translated by  $r^{\bar{a}}(t)$  relative to the origin of the global inertial frame; this is the position of the body's center-of-mass.

The equations of fluid dynamics are reviewed, for example, in Secs. 1 and 2 of Ref. [15]. A perfect fluid is governed by (i) the continuity equation

$$\partial_t \rho + \partial_{\bar{a}}(\rho v^{\bar{a}}) = 0, \quad (2.1)$$

in which  $\rho$  is the mass density and  $v^{\bar{a}}$  the velocity field; (ii) Euler's equation

$$\rho \frac{dv^{\bar{a}}}{dt} = -\partial_{\bar{a}} p + \rho \partial_{\bar{a}} U - \rho \ddot{r}_{\bar{a}}, \quad (2.2)$$

in which  $dv^{\bar{a}}/dt = \partial_t v^{\bar{a}} + v^{\bar{b}} \partial_{\bar{b}} v^{\bar{a}}$  is the advective time derivative,  $p$  the pressure,  $U$  the gravitational potential, and the last term is an inertial force density associated with the translation of the frame  $x^{\bar{a}}$  relative to the global inertial frame; (iii) Poisson's equation

$$\nabla^2 U = -4\pi G \rho \quad (2.3)$$

for the gravitational potential; and (iv) an equation of state that relates the pressure to the density.

In the case of an incompressible fluid,  $\rho = \text{constant}$  and the fluid equations become

$$\partial_{\bar{a}} v^{\bar{a}} = 0 \quad (2.4)$$

and

$$\frac{dv^{\bar{a}}}{dt} = \partial_{\bar{a}} \left( -p/\rho + U - \ddot{r}_{\bar{b}} x^{\bar{b}} \right). \quad (2.5)$$

In this case the pressure is unrelated to the density.

It is useful to decompose the Newtonian potential as  $U = U_{\text{body}} + U_{\text{ext}}$ , in which  $U_{\text{body}}$  is associated with the body and  $U_{\text{ext}}$  is produced by the external objects. We also decompose the external potential as  $U_{\text{ext}}(t, x^{\bar{a}}) = U_0(t) + g_{\bar{a}}(t)x^{\bar{a}} + U_{\text{tidal}}(t, x^{\bar{a}})$ , in which  $U_0(t) := U_{\text{ext}}(t, r^{\bar{a}})$  and  $g_{\bar{a}} := \partial_{\bar{a}} U_{\text{ext}}(t, r^{\bar{a}})$ ; this provides a definition for the tidal potential  $U_{\text{tidal}}$ . Because the body moves according to  $\ddot{r}_{\bar{a}} = g_{\bar{a}}$ , the final form of Euler's equation is

$$\frac{dv^{\bar{a}}}{dt} = \partial_{\bar{a}} \left( -p/\rho + U_{\text{body}} + U_{\text{tidal}} \right). \quad (2.6)$$

### C. Fluid equations; rotating frame

The transformation from the frame  $x^{\bar{a}} = [\bar{x}, \bar{y}, \bar{z}]$  to the rotating frame  $x^j = [x, y, z]$  is described by

$$x = \bar{x} \cos \Omega_H t + \bar{y} \sin \Omega_H t, \quad (2.7a)$$

$$y = -\bar{x} \sin \Omega_H t + \bar{y} \cos \Omega_H t, \quad (2.7b)$$

$$z = \bar{z}, \quad (2.7c)$$

if we take the rotation axis to coincide with the  $z$ -axis. The transformation can be written more formally as

$$x^j = \Lambda^j_{\bar{a}} x^{\bar{a}}. \quad (2.8)$$

The velocity vector transforms as

$$v^x = v^{\bar{x}} \cos \Omega_H t + v^{\bar{y}} \sin \Omega_H t + \Omega_H (-\bar{x} \sin \Omega_H t + \bar{y} \cos \Omega_H t), \quad (2.9a)$$

$$v^y = -v^{\bar{x}} \sin \Omega_H t + v^{\bar{y}} \cos \Omega_H t - \Omega_H (\bar{x} \cos \Omega_H t + \bar{y} \sin \Omega_H t), \quad (2.9b)$$

$$v^z = v^{\bar{z}}, \quad (2.9c)$$

which can be written formally as

$$v^j = \Lambda^j_{\bar{a}} (v^{\bar{a}} + \Delta v^{\bar{a}}), \quad (2.10)$$

where

$$\Delta v^{\bar{a}} := \Gamma^{\bar{a}}_{\bar{b}} \dot{x}^{\bar{b}} \quad (2.11)$$

accounts for the time-dependence of the transformation. The matrix  $\Gamma^{\bar{a}}_{\bar{b}}$  is formally defined by  $\Gamma^{\bar{a}}_{\bar{b}} = \Lambda^{\bar{a}}_{\bar{j}} \dot{\Lambda}^j_{\bar{b}}$ , where  $\Lambda^{\bar{a}}_{\bar{j}}$  is the inverse of the matrix  $\Lambda^j_{\bar{a}}$ . Calculation reveals that

$$\Gamma^{\bar{a}}_{\bar{b}} = \Omega_H \epsilon^{\bar{a}}_{\bar{b}\bar{c}} s^{\bar{c}}, \quad (2.12)$$

where  $\epsilon_{\bar{a}\bar{b}\bar{c}}$  is the permutation symbol and the vector  $s^{\bar{c}} = [0, 0, 1]$  denotes the direction of the rotation axis.

Making the substitution in Euler's equation produces additional terms that originate from the noninertial nature of the transformation. The first collection of additional terms are linear in both  $\Omega_H$  and  $v^j$ , and are responsible for the Coriolis effect; these we neglect, according to the assumptions formulated in Sec. II A. The second collection of terms are quadratic in  $\Omega_H$  and independent of  $v^j$ , and give rise to centrifugal repulsion; these we keep. To simplify the equation further we neglect all terms quadratic in  $v^j$  and the inertial term  $\partial_t v^j$ , again in accordance with the stated assumptions.

As a result of these substitutions we find that the fluid equations in the rotating frame are  $\partial_j v^j = 0$ ,

$$\partial_j (-p/\rho + U_{\text{body}} + U_{\text{tidal}} + \Omega_H^2 C) = 0, \quad (2.13)$$

and the gravitational potentials are determined by  $\nabla^2 U_{\text{body}} = -4\pi G\rho$  and  $\nabla^2 U_{\text{tidal}} = 0$ . Here  $\Omega_H^2 C = \frac{1}{2}\Omega_H^2(x^2 + y^2)$  is the centrifugal potential. It is useful to express  $C$  as

$$C = \frac{1}{6}r^2 + \frac{1}{2}C_{jk}x^j x^k, \quad (2.14)$$

where

$$C_{jk} = \frac{1}{3}\delta_{jk} - s_j s_k \quad (2.15)$$

is a symmetric-tracefree (STF) tensor. The first term in  $C$ 's decomposition is spherically symmetric; the second has a quadrupolar ( $\ell = 2$ ) structure.

#### D. Tidal field

The gravitational potential was decomposed as

$$U = U_{\text{body}} + U_{\text{tidal}}, \quad (2.16)$$

with  $U_{\text{body}}$  representing the potential of the (deformed) body, and  $U_{\text{tidal}}$  representing the tidal potential. To leading order in a Taylor expansion of the external potential about the body's center-of-mass, this is

$$U_{\text{tidal}} = -\frac{1}{2}\mathcal{E}_{jk}(t)x^j x^k, \quad (2.17)$$

where the tidal moment  $\mathcal{E}_{jk}$  is a time-dependent STF tensor that does not depend on spatial position. The tidal potential also has a quadrupolar structure.

A specific example of a tidal field is one produced by an external object of mass  $M'$  on a circular orbit of radius  $b$  around the reference body. If we let  $m^j$  be a unit vector that points in the direction of the external object (from the body's center-of-mass), a simple computation reveals that the tidal moment is given by

$$\mathcal{E}_{jk}(t) = -\frac{GM'}{3b^3}\left(m_j m_k - \frac{1}{3}\delta_{jk}\right). \quad (2.18)$$

If we take the external object to move in the equatorial plane of the rotating body, the unit vector is given by

$$m^j(t) = [\cos(\Omega - \Omega_H)t, \sin(\Omega - \Omega_H)t, 0] \quad (2.19)$$

in the rotating frame, where  $\Omega$  is the orbital angular velocity.

#### E. Body deformation

The tidal forces and body rotation produce a deformation in the body's shape, which can be described by

$$R = a(1 + e_{jk}n^j n^k), \quad (2.20)$$

where  $n^j = x^j/r$  is the radial unit vector. The equation  $r = R(n^j)$  describes the position of the body's deformed surface. We let the deformation assume a quadrupolar shape, in accordance with the fact that each deforming force is quadrupolar in nature;  $e_{jk}$  is a STF tensor that must be determined by solving the fluid equations.

#### F. Body potential

The body potential is given by

$$U_{\text{body,out}} = \frac{GM}{r} + \frac{3}{2}GQ_{jk}\frac{x^j x^k}{r^5} \quad (2.21)$$

outside the body, and

$$U_{\text{body,in}} = U_0(r) + U_{jk}x^j x^k \quad (2.22)$$

inside the body. In each case the deviation from a spherical potential is measured by a STF tensor; we have the quadrupole-moment tensor  $Q_{jk}$  outside, and  $U_{jk}$  inside.

The body mass is related to the density by

$$M = \frac{4\pi}{3}\rho a^3. \quad (2.23)$$

The quadrupole-moment tensor is determined by evaluating  $Q^{jk} = \int \rho x^{(jk)} d^3x$ , where  $x^{(jk)} := x^j x^k - \frac{1}{3}r^2\delta^{jk}$ ; a quick computation returns

$$Q_{jk} = \frac{2}{5}Ma^2 e_{jk}. \quad (2.24)$$

The internal potential  $U_0$  is obtained by solving Poisson's equation for a spherical configuration of constant density  $\rho$  and radius  $a$ ; the result is

$$U_0(r) = \frac{GM}{2a}(3 - r^2/a^2). \quad (2.25)$$

And finally,  $U_{jk}$  is obtained by demanding continuity of the body potential across the deformed surface; this yields

$$U_{jk} = \frac{3}{5}\frac{GM}{a^3}e_{jk}. \quad (2.26)$$

### G. Pressure

The pressure field inside the body is expressed as

$$p/\rho = p_0(r)/\rho + p_{jk}x^jx^k, \quad (2.27)$$

where  $p_{jk}$  is another STF tensor. This form is motivated by the fact that according to Eq. (2.13), the pressure must satisfy the Poisson equation  $\nabla^2(p/\rho) = -4\pi G\rho + \Omega_H^2$ . Because the right-hand side is spherically symmetric,  $p_0/\rho$  must satisfy this equation by itself, and the additional term must be a solution to Laplace's equation; a quadrupolar structure necessarily selects the assumed form  $p_{jk}x^jx^k$ .

The spherically-symmetric component of the pressure is easily obtained by integrating Poisson's equation; we get

$$p_0(r)/\rho = \frac{GM}{2a} \left(1 - \frac{\Omega_H^2}{4\pi G\rho}\right) \left(1 - r^2/a^2\right) \quad (2.28)$$

when we demand that the unperturbed pressure vanish on the unperturbed surface  $r = a$ . An expression for  $p_{jk}$  is obtained when we demand that the perturbed pressure vanish on the perturbed surface. If we let  $\xi := R - a = ae_{jk}n^jn^k$  describe the deformation, we have that  $0 = p(a + \xi) = p_0(a) + ap'_0(a)e_{jk}n^jn^k + a^2p_{jk}n^jn^k$ , in which a prime indicates differentiation with respect to  $r$ . The computation returns

$$p_{jk} = \frac{GM}{a^3} e_{jk} \quad (2.29)$$

after neglecting the correction of order  $\Omega_H^2/(4\pi G\rho)$  in  $p_0(r)$ .

### H. Solution

Euler's equation (2.13) produces the constraint

$$-p_{jk} + U_{jk} - \frac{1}{2}\mathcal{E}_{jk} + \frac{1}{2}\Omega_H^2 C_{jk} = 0 \quad (2.30)$$

when we insert Eqs. (2.14), (2.17), (2.22), and (2.27) into it. Since  $p_{jk}$  and  $U_{jk}$  are both proportional to the deformation tensor  $e_{jk}$ , we quickly deduce that

$$e_{jk} = \frac{5}{4} \frac{a^3}{GM} (\Omega_H^2 C_{jk} - \mathcal{E}_{jk}). \quad (2.31)$$

This result informs us that the deformation is a combination of rotational and tidal effects, as expected. We also have

$$p_{jk} = \frac{5}{4} (\Omega_H^2 C_{jk} - \mathcal{E}_{jk}), \quad (2.32)$$

$$U_{jk} = \frac{3}{4} (\Omega_H^2 C_{jk} - \mathcal{E}_{jk}), \quad (2.33)$$

and the quadrupole-moment tensor is

$$Q_{jk} = \frac{1}{2} \frac{a^5}{G} (\Omega_H^2 C_{jk} - \mathcal{E}_{jk}). \quad (2.34)$$

## III. TIDAL INTERACTION OF A NEWTONIAN BODY; VISCOUS CASE

We now incorporate viscosity into the fluid model. The fluid's shear viscosity is measured by  $\nu$ , the coefficient of kinematic viscosity, which we assume to be uniform within the fluid.

### A. Scales and assumptions

We continue to deal with the scaling quantities introduced in Sec. II A, and with one exception, we make the same assumptions regarding them as we did in the preceding section. The only change concerns the time scale of the problem, which previously was provided by the orbital dynamics as viewed in the rotating frame; the only relevant time scale was  $|\Omega - \Omega_H|^{-1}$ . The addition of viscosity introduces another relevant time scale, namely the delay between the action of the applied tidal force and the body's response. As we shall see, the time scale for the viscous delay is  $\tau \sim a\nu/(GM)$ , and we assume that this is much shorter than the orbital time scale:  $\tau \ll |\Omega - \Omega_H|^{-1}$ ; this inequality constrains the size of the kinematic viscosity.

The change of time scale has repercussions on the velocity scale. It is now given by  $v \sim \xi/\tau$ , where  $\xi \sim a^4\mathcal{E}/(GM)$  is still the length scale associated with fluid displacements. To neglect the Coriolis effect with still demand that  $v \ll a\Omega_H$ , but the inequality now translates to  $\mathcal{E} \ll (GM/a^3)(\Omega_H\tau)$ , or  $(a/b)^3 \ll (M/M')(\Omega_H\tau)$ ; since  $\Omega_H\tau \ll 1$  we find that the bodies must be even more widely separated. To neglect the inertial term in the fluid equation we now demand that  $v/\tau \ll \mathcal{E}a$ , and this translates to  $\tau \gg \sqrt{a^3/(GM)}$ , another constraint on the kinematic viscosity. (The constraints are compatible with each other whenever  $|\Omega - \Omega_H| \ll \tau^{-1} \ll \Omega_K$ , where  $\Omega_K = \sqrt{GM/a^3}$  is the Keplerian angular velocity of a particle at the body's surface.)

### B. Fluid equations

Under the stated assumptions, the fluid equations in the rotating frame are now (see, for example, Sec. 15 of Ref. [15])

$$\partial_j v^j = 0 \quad (3.1)$$

and

$$\partial_j (-p/\rho + U_{\text{body}} + U_{\text{tidal}} + C) + \nu \nabla^2 v^j = 0, \quad (3.2)$$

with the last term giving rise to all effects associated with the viscosity. The centrifugal potential is still given by

$$C = \frac{1}{6} \Omega_H^2 r^2 + \frac{1}{2} \Omega_H^2 C_{jk} x^j x^k, \quad (3.3)$$

and the internal gravitational potential is still given by

$$U_{\text{body}} + U_{\text{tidal}} = U_0(r) + U_{jk}x^jx^k - \frac{1}{2}\mathcal{E}_{jk}x^jx^k, \quad (3.4)$$

with  $U_0(r) = GM(3 - r^2/a^2)/(2a)$  and  $U_{jk} = 3GMe_{jk}/(5a^3)$ . The body deformation is still described by  $R = a(1 + e_{jk}n^jn^k)$ , and the quadrupole moment is still  $Q_{jk} = \frac{2}{5}Ma^2e_{jk}$ .

The presence of viscosity affects the nature of the boundary conditions at the body's surface. Instead of the no-pressure condition imposed previously, we must now impose the no-normal-stress condition

$$pn_j - \rho\nu(\partial_jv_k + \partial_kv_j)n^k = 0 \quad (3.5)$$

at  $r = R$ . In addition, we must relate the normal component of the velocity vector to the rate of change of the surface displacement:

$$v_jn^j\Big|_{r=a} = \partial_t R. \quad (3.6)$$

Unlike the preceding equation (which involves the unperturbed piece of the pressure), this equation can be formulated at  $r = a$  instead of  $r = R$ , because it involves perturbed quantities only.

Because the pressure still satisfies the Poisson equation  $\nabla^2(p/\rho) = -4\pi G\rho + \Omega_H^2$ , we may still express it as

$$p/\rho = p_0(r)/\rho + p_{jk}x^jx^k. \quad (3.7)$$

The expression for  $p_0(r)$  is unchanged, but  $p_{jk}$  is no longer equal to  $GMe_{jk}/a^3$ ; its new value will be determined by the new boundary conditions. Using the available information, we find that these become

$$\nu(\partial_jv_k + \partial_kv_j)n^k = a^2n_j\left(p_{kn} - \frac{GM}{a^3}e_{kn}\right)n^kn^n \quad (3.8)$$

and

$$v_jn^j = a\dot{e}_{jk}n^jn^k. \quad (3.9)$$

Both equations are now formulated at  $r = a$ .

### C. Velocity field

At this stage the unknowns are the velocity field  $v^j$ , the displacement tensor  $e_{jk}$ , and the pressure tensor  $p_{jk}$ ; these must all be determined in terms of  $C_{jk}$  and  $\mathcal{E}_{jk}$ . The velocity field obeys a Poisson equation with a quadrupolar source term. The solution must be a  $\ell = 2$  vectorial harmonic, and it can therefore be expressed as a linear superposition of terms  $r^2V_{jk}x^k$ ,  $V_{kn}x^jx^kx^n$ , and  $V_{jk}x^k$ , with  $V_{jk}$  a time-dependent STF tensor; the superposition is constrained by the divergence-free condition  $\partial_jv^j = 0$ . The boundary condition of Eq. (3.9) relates  $V_{jk}$  to  $\dot{e}_{jk}$ , and Eq. (3.8) determines  $p_{jk}$  in terms of  $e_{jk}$  and its time derivative. Finally, the Navier-Stokes equation

(3.2) gives rise to a differential equation for the displacement tensor  $e_{jk}$ .

After going through these manipulations and sorting out the algebra, we arrive at the differential equation

$$\tau\dot{e}_{jk} + e_{jk} = \frac{5}{4}\frac{a^3}{GM}(\Omega_H^2C_{jk} - \mathcal{E}_{jk}), \quad (3.10)$$

where

$$\tau := \frac{19}{2}\frac{a\nu}{GM} \quad (3.11)$$

is the viscous delay. We also obtain the pressure tensor

$$p_{jk} = \frac{GM}{a^3}\left(e_{jk} - \frac{4\tau}{95}\dot{e}_{jk}\right) \quad (3.12)$$

and the velocity field

$$v^j = -\frac{1}{5a^2}\left[(5r^2 - 8a^2)\dot{e}_{jk}x^k - 2\dot{e}_{kn}x^jx^kx^n\right]. \quad (3.13)$$

### D. Solution

The solution to the differential equation  $\tau\dot{e} + e = f$  is

$$e(t) = e^{-t/\tau}\left[e(0) + \frac{1}{\tau}\int_0^t e^{t'/\tau}f(t')dt'\right]. \quad (3.14)$$

Two integrations by parts bring this to the form

$$\begin{aligned} e(t) &= [e(0) - f(0) + \tau\dot{f}(0)]e^{-t/\tau} + f(t) - \tau\dot{f}(t) \\ &\quad + \tau e^{-t/\tau}\int_0^t e^{t'/\tau}\ddot{f}(t')dt'. \end{aligned} \quad (3.15)$$

At times  $t \gg \tau$  we may neglect the first group of terms, and we may also neglect the integral if  $f(t)$  varies on a time scale that is long compared with  $\tau$ . Under these circumstances the solution is

$$e(t) = f(t) - \tau\dot{f}(t) + O[e(0)e^{-t/\tau}] + O[\tau^2\ddot{f}(t)]. \quad (3.16)$$

Within the stated error, this is  $e(t) = f(t - \tau)$ . We see that the displacement at time  $t$  is related to the behavior of the driving force at the earlier time  $t - \tau$ , and the delay is given precisely by  $\tau$ . The viscosity therefore introduces a phase lag between the applied force and the response, and as we shall see, this dephasing turns out to be crucial for the physics of tidal work, torque, and heating.

In our context the role of the driving force  $f$  is played by the tidal moment  $\mathcal{E}_{jk}$ , which varies on a time scale of order  $|\Omega - \Omega_H|^{-1}$ . We already have stated our assumption that this time scale is much longer than  $\tau$ . In these circumstances, and ignoring the transients that decay exponentially, the solution to Eq. (3.10) is

$$e_{jk} = \frac{5}{4}\frac{a^3}{GM}(\Omega_H^2C_{jk} - \mathcal{E}_{jk} + \tau\dot{\mathcal{E}}_{jk}). \quad (3.17)$$



With this we find that

$$p_{jk} = \frac{5}{4} \left( \Omega_H^2 C_{jk} - \mathcal{E}_{jk} + \frac{99\tau}{95} \dot{\mathcal{E}}_{jk} \right), \quad (3.18)$$

$$U_{jk} = \frac{3}{4} \left( \Omega_H^2 C_{jk} - \mathcal{E}_{jk} + \tau \dot{\mathcal{E}}_{jk} \right), \quad (3.19)$$

and

$$v^j = \frac{a}{4GM} \left[ (5r^2 - 8a^2) \dot{\mathcal{E}}_{jk} x^k - 2\dot{\mathcal{E}}_{kn} x^j x^k x^n \right]. \quad (3.20)$$

The quadrupole moment tensor is now

$$Q_{jk} = \frac{a^5}{2G} \left( \Omega_H^2 C_{jk} - \mathcal{E}_{jk} + \tau \dot{\mathcal{E}}_{jk} \right). \quad (3.21)$$

### E. Transformation to the nonrotating frame

For later purposes we transform our results from the rotating frame  $x^j$  back to the nonrotating frame  $x^{\bar{a}}$ ; the transformation is described by  $x^{\bar{a}} = \Lambda_{\bar{a}}^j x^j$ , and it is the inverse of the one considered in Sec. II C.

As we have seen, the velocity field transforms as

$$v^{\bar{a}} = \Lambda_{\bar{a}}^j (v^j + \Delta v^j), \quad (3.22)$$

where

$$\Delta v^j := \Gamma_k^j x^k \quad (3.23)$$

accounts for the time-dependence of the transformation; here

$$\Gamma_k^j := \Lambda_{\bar{a}}^j \dot{\Lambda}_{\bar{a}}^{\bar{k}} = -\Omega_H \epsilon_{kn}^j s^n, \quad (3.24)$$

with  $s^n = [0, 0, 1]$  denoting the direction of the rotation axis.

In the nonrotating frame the tidal moment is given by

$$\mathcal{E}_{\bar{a}\bar{b}} = \Lambda_{\bar{a}}^j \Lambda_{\bar{b}}^k \mathcal{E}_{jk}. \quad (3.25)$$

The transformation of its time derivative, however, must account for the time-dependence of the transformation. Here we have

$$\dot{\mathcal{E}}_{\bar{a}\bar{b}} = \Lambda_{\bar{a}}^j \Lambda_{\bar{b}}^k (\dot{\mathcal{E}}_{jk} - \Delta \dot{\mathcal{E}}_{jk}) = \Lambda_{\bar{a}}^j \Lambda_{\bar{b}}^k \dot{\mathcal{E}}_{jk} - \Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}}, \quad (3.26)$$

with

$$\Delta \dot{\mathcal{E}}_{jk} := \Gamma_j^n \mathcal{E}_{nk} + \Gamma_k^n \mathcal{E}_{jn} = 2\Omega_H \epsilon_{pq(j} \mathcal{E}_{k)}^p s^q \quad (3.27)$$

and

$$\Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}} := \Lambda_{\bar{a}}^j \Lambda_{\bar{b}}^k \Delta \dot{\mathcal{E}}_{jk} = 2\Omega_H \epsilon_{\bar{c}\bar{d}(\bar{a}} \mathcal{E}_{\bar{b})}^{\bar{c}} s^{\bar{d}}. \quad (3.28)$$

The transformation of the quadrupole-moment tensor is  $Q_{\bar{a}\bar{b}} = \Lambda_{\bar{a}}^j \Lambda_{\bar{b}}^k Q_{jk}$ . With Eqs. (3.21) and (3.26), this is

$$Q_{\bar{a}\bar{b}} = \frac{a^5}{2G} \left( \Omega_H^2 C_{\bar{a}\bar{b}} - \mathcal{E}_{\bar{a}\bar{b}} + \tau \dot{\mathcal{E}}_{\bar{a}\bar{b}} + \tau \Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}} \right), \quad (3.29)$$

with the last term accounting for the effect of the body's rotation on the tidally-induced quadrupole moment.

### F. Example

To illustrate the way in which the preceding (simple, but potentially confusing) tensorial transformations work, we examine the case in which the rotating-frame tidal moment is given by Eq. (2.18),

$$\mathcal{E}_{jk} = -\frac{GM'}{3b^3} \left( m_j m_k - \frac{1}{3} \delta_{jk} \right), \quad (3.30)$$

with  $m^j = [\cos(\Omega - \Omega_H)t, \sin(\Omega - \Omega_H)t, 0]$ . The transformed tidal moment is

$$\mathcal{E}_{\bar{a}\bar{b}} = -\frac{GM'}{3b^3} \left( m_{\bar{a}} m_{\bar{b}} - \frac{1}{3} \delta_{\bar{a}\bar{b}} \right), \quad (3.31)$$

with  $m^{\bar{a}} = [\cos \Omega t, \sin \Omega t, 0]$ . The time-derivative of the rotating-frame tidal field is

$$\dot{\mathcal{E}}_{jk} = -\frac{GM'}{3b^3} (\Omega - \Omega_H) (m_j \phi_k + \phi_j m_k), \quad (3.32)$$

where  $\phi^j := [-\sin(\Omega - \Omega_H)t, \cos(\Omega - \Omega_H)t, 0]$ . From this we find that

$$\Lambda_{\bar{a}}^j \Lambda_{\bar{b}}^k \dot{\mathcal{E}}_{jk} = -\frac{GM'}{3b^3} (\Omega - \Omega_H) (m_{\bar{a}} \phi_{\bar{b}} + \phi_{\bar{a}} m_{\bar{b}}), \quad (3.33)$$

with  $\phi^{\bar{a}} = [-\sin \Omega t, \cos \Omega t, 0]$ . On the other hand,

$$\Delta \dot{\mathcal{E}}_{jk} = \frac{GM'}{3b^3} \Omega_H (m_j \phi_k + \phi_j m_k) \quad (3.34)$$

and

$$\Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}} = \frac{GM'}{3b^3} \Omega_H (m_{\bar{a}} \phi_{\bar{b}} + \phi_{\bar{a}} m_{\bar{b}}). \quad (3.35)$$

We see that these pieces add up correctly to produce the expected

$$\dot{\mathcal{E}}_{\bar{a}\bar{b}} = -\frac{GM'}{3b^3} \Omega (m_{\bar{a}} \phi_{\bar{b}} + \phi_{\bar{a}} m_{\bar{b}}). \quad (3.36)$$

The combination of terms that appears in  $Q_{\bar{a}\bar{b}}$ , however, is

$$\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}} = -\frac{GM'}{3b^3} (\Omega - \Omega_H) (m_{\bar{a}} \phi_{\bar{b}} + \phi_{\bar{a}} m_{\bar{b}}), \quad (3.37)$$

and we see that this vanishes when the external object is corotating with the body.

## IV. TIDAL WORK, TORQUE, AND HEATING

### A. Tidal work

The rate at which the tidal forces do work on the rotating body is calculated as

$$\dot{W} = \int (-\rho \mathcal{E}_{\bar{a}\bar{b}} x^{\bar{b}}) v^{\bar{a}} d^3 \bar{x}.$$

This is the integral of the tidal force density  $-\rho\mathcal{E}_{\bar{a}\bar{b}}x^{\bar{b}}$  times  $v^{\bar{a}}$ , the rate of fluid displacement. This is equal to

$$\dot{W} = -\frac{1}{2}\mathcal{E}_{\bar{a}\bar{b}}\dot{Q}^{\bar{a}\bar{b}}, \quad (4.1)$$

or

$$\dot{W} = \frac{1}{2}Q_{\bar{a}\bar{b}}\dot{\mathcal{E}}^{\bar{a}\bar{b}} - \frac{1}{2}\frac{d}{dt}\left(Q_{\bar{a}\bar{b}}\mathcal{E}^{\bar{a}\bar{b}}\right).$$

The total derivative corresponds to a change of a state function, and we shall ignore such (reversible) changes in this analysis. Our final expression for the rate of tidal work shall be

$$\dot{W} = \frac{1}{2}Q_{\bar{a}\bar{b}}\dot{\mathcal{E}}^{\bar{a}\bar{b}}. \quad (4.2)$$

If we insert Eq. (3.29) in this we find that the  $\Omega_H^2 C_{\bar{a}\bar{b}}$  and  $\mathcal{E}_{\bar{a}\bar{b}}$  terms contribute total derivatives, which we continue to ignore. The only nontrivial contributions come from the terms proportional to  $\tau$ ; we get

$$\dot{W} = \frac{a^5\tau}{4G}\dot{\mathcal{E}}^{\bar{a}\bar{b}}(\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta\dot{\mathcal{E}}_{\bar{a}\bar{b}}). \quad (4.3)$$

We recall that

$$\Delta\dot{\mathcal{E}}_{\bar{a}\bar{b}} = 2\Omega_H\epsilon_{\bar{a}\bar{d}(\bar{a}}\mathcal{E}_{\bar{b})}^{\bar{c}}s^{\bar{d}} \quad (4.4)$$

and

$$\tau = \frac{19}{2}\frac{a\nu}{GM}. \quad (4.5)$$

The equation reveals that there is no net tidal work in the absence of viscosity. The dephasing between  $Q_{\bar{a}\bar{b}}$  and  $\mathcal{E}_{\bar{a}\bar{b}}$  created by the viscous delay is therefore an essential piece of the physics. In the absence of viscosity, the fluid displacement is always in phase with the applied force, and this leads to a reversible transfer of gravitational energy between the body and the tidal field; the work done necessarily averages to zero.

For an external object of mass  $M'$  on a circular orbit of radius  $b$  and angular velocity  $\Omega$ , the rate of tidal work is

$$\dot{W} = \frac{a^5\tau}{18G}\frac{(GM')^2}{b^6}\Omega(\Omega - \Omega_H). \quad (4.6)$$

There is no work done if the external object is corotating with the body.

### B. Tidal torque

The total torque exerted by the tidal forces is

$$\tau_{\bar{a}} = \int \epsilon_{\bar{a}\bar{b}\bar{c}}x^{\bar{b}}(-\rho\mathcal{E}_{\bar{c}}^{\bar{d}}x^{\bar{d}})d^3\bar{x},$$

and this is equal to  $\dot{J}_{\bar{a}}$ , the rate of change of the body's angular-momentum vector. Evaluation of the integral gives

$$\dot{J}_{\bar{a}} = -\epsilon_{\bar{a}\bar{b}\bar{c}}Q_{\bar{p}}^{\bar{b}}\mathcal{E}^{\bar{p}\bar{c}}. \quad (4.7)$$

The angular-momentum vector can be decomposed as  $J^{\bar{a}} = Js^{\bar{a}}$ , in terms of a magnitude  $J$  and a direction  $s^{\bar{a}}$ . The change in angular momentum is then expressed as  $\dot{J}^{\bar{a}} = \dot{J}s^{\bar{a}} + Js^{\bar{a}}$ . The first term describes the change in the body's rate of rotation, while the second describes precessional effects. We are interested here in the rate of change of  $J$ , and therefore select the projection of  $\dot{J}^{\bar{a}}$  in the direction of  $s^{\bar{a}}$ . This is

$$\dot{J} = -\epsilon_{\bar{a}\bar{b}\bar{c}}Q_{\bar{p}}^{\bar{a}}\mathcal{E}^{\bar{p}\bar{b}}s^{\bar{c}}. \quad (4.8)$$

When we insert Eq. (3.29) in this we find once more that the  $\Omega_H^2 C_{\bar{a}\bar{b}}$  and  $\mathcal{E}_{\bar{a}\bar{b}}$  terms make no contribution. We obtain

$$\dot{J} = -\frac{a^5\tau}{2G}\epsilon_{\bar{p}\bar{q}}^{\bar{a}}\mathcal{E}^{\bar{p}\bar{b}}s^{\bar{q}}(\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta\dot{\mathcal{E}}_{\bar{a}\bar{b}}). \quad (4.9)$$

We relate this to the rate of change of the body's rotational energy  $E_{\text{rot}} = \frac{1}{2}I\Omega_H^2$ , where  $I$  is the body's moment of inertia. With  $J = I\Omega_H$  we find that  $\dot{E}_{\text{rot}} = \Omega_H\dot{J}$ . Taking into account Eq. (4.4), Eq. (4.9) becomes

$$\dot{E}_{\text{rot}} = \Omega_H\dot{J} = -\frac{a^5\tau}{4G}\Delta\dot{\mathcal{E}}^{\bar{a}\bar{b}}(\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta\dot{\mathcal{E}}_{\bar{a}\bar{b}}). \quad (4.10)$$

Again we find that the effect vanishes in the absence of viscosity: It is the misalignment between the directions of the tidal forces and tidal bulge that allows the forces to exert a torque on the body.

For an external object of mass  $M'$  on a circular orbit of radius  $b$  and angular velocity  $\Omega$ , the rate of change of the body's rotational energy is

$$\dot{E}_{\text{rot}} = \Omega_H\dot{J} = \frac{a^5\tau}{18G}\frac{(GM')^2}{b^6}\Omega_H(\Omega - \Omega_H). \quad (4.11)$$

There is no change if the external object is corotating with the body.

### C. Heat dissipation

The rate at which heat is dissipated within the fluid is given by (see, for example, Sec. 16 of Ref. [15])

$$\dot{Q} = \frac{1}{2}\rho\nu\int(\partial_{\bar{a}}v_{\bar{b}} + \partial_{\bar{b}}v_{\bar{a}})(\partial^{\bar{a}}v^{\bar{b}} + \partial^{\bar{b}}v^{\bar{a}})d^3\bar{x}. \quad (4.12)$$

The velocity field in the norotating frame is obtained by transforming Eq. (3.20) using the rules spelled out in Sec. III E. We have

$$\begin{aligned} v^{\bar{a}} = & \Omega_H\epsilon_{\bar{b}\bar{c}}^{\bar{a}}s^{\bar{b}}x^{\bar{c}} + \frac{a}{4GM}\left[(5\bar{r}^2 - 8a^2)(\dot{\mathcal{E}}_{\bar{b}}^{\bar{a}} + \Delta\dot{\mathcal{E}}_{\bar{b}}^{\bar{a}})x^{\bar{b}} \right. \\ & \left. - 2(\dot{\mathcal{E}}_{\bar{b}\bar{c}} + \Delta\dot{\mathcal{E}}_{\bar{b}\bar{c}})x^{\bar{a}}x^{\bar{b}}x^{\bar{c}}\right], \end{aligned} \quad (4.13)$$

and this can be inserted within our expression for  $\dot{Q}$ .

A straightforward computation returns

$$\dot{Q} = \frac{a^5 \tau}{4G} (\dot{\mathcal{E}}^{\bar{a}\bar{b}} + \Delta \dot{\mathcal{E}}^{\bar{a}\bar{b}}) (\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}}). \quad (4.14)$$

Notice that this is positive-definite, as it should be.

For an external object of mass  $M'$  on a circular orbit of radius  $b$  and angular velocity  $\Omega$ , the rate at which heat is dissipated within the fluid is

$$\dot{Q} = \frac{a^5 \tau}{18G} \frac{(GM')^2}{b^6} (\Omega - \Omega_H)^2. \quad (4.15)$$

#### D. Energy balance

The work done by the tidal forces is used partially to increase the body's rotational energy, and is partially dissipated into heat by the fluid's viscosity. Energy balance requires that

$$\dot{W} = \dot{Q} + \Omega_H \dot{J}, \quad (4.16)$$

and we see from Eqs. (4.3), (4.10), and (4.14) that the equation is indeed satisfied.

#### E. Generalization to arbitrary internal structure and correspondence with black-hole physics

The results derived previously apply specifically to an incompressible fluid. It is easy, however, to generalize them so that they apply to a fluid body with an arbitrary internal structure (generated by an arbitrary equation of state). The trick is to preserve the main structure of each equation, including the scaling with each dimensionful quantity, but to insert various dimensionless coefficients where appropriate. These coefficients are called “Love numbers,” and their role is precisely to incorporate the details of the body's internal structure. This strategy is viable so long as one is interested only in the structure and scalings of the main equations, and one is satisfied not to know the exact value of each Love number. A detailed computation of the Love numbers for a selected equation of state would require the hard work of repeating the analysis of fluid perturbations for this equation of state.

The first result that can be generalized in this way is Eq. (3.21), which relates  $Q_{jk}$ , the quadrupole moment of the body's mass distribution, to  $\mathcal{E}_{jk}$ , the quadrupole moment of the tidal field. The generalization appears in Eq. (1.5); it involves a rotational Love number  $n_2$ , a tidal Love number  $k_2$ , and a viscous delay  $\tau$  that is known to scale as  $a\nu/(GM)$  with the body radius  $a$ , its mass  $M$ ,

and its averaged kinetic viscosity  $\nu$ . For an incompressible fluid with a uniform viscosity,  $n_2 = k_2 = \frac{3}{4}$  and  $\tau = \frac{19}{2} a\nu/(GM)$ . [In the generalization I chose to promote  $\tau$  to a dimensionful “Love quantity.” An alternative would have been to define a new Love number  $v_2$  and to generalize  $\tau$  as  $v_2 a\nu/(GM)$ . The former strategy is more useful for the purpose of establishing a correspondence with the black-hole results.] Similarly, the generalization of Eq. (3.29) appears in Eq. (1.6).

The expressions obtained for  $\dot{W}$  and  $\dot{J}$  — see Eqs. (4.3) and (4.10), respectively — follow directly from the basic definitions of Eqs. (4.2), (4.8) and the relation of Eq. (3.29) between  $Q_{\bar{a}\bar{b}}$  and  $\mathcal{E}_{\bar{a}\bar{b}}$ . The definitions continue to apply even after Eq. (3.29) is generalized to Eq. (1.6), and we arrive at

$$\dot{W} = \frac{(k_2 \tau) a^5}{3G} \dot{\mathcal{E}}^{\bar{a}\bar{b}} (\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}}) \quad (4.17)$$

and

$$\Omega_H \dot{J} = -\frac{(k_2 \tau) a^5}{3G} \Delta \dot{\mathcal{E}}^{\bar{a}\bar{b}} (\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}}). \quad (4.18)$$

These are Eqs. (1.8) and (1.9), respectively (in which the overbar is removed from the indices to simplify the notation, and  $G$  is set equal to unity). We can then invoke the energy-balance equation (4.16) to motivate the generalized form of Eq. (4.14),

$$\dot{Q} = \frac{(k_2 \tau) a^5}{3G} (\dot{\mathcal{E}}^{\bar{a}\bar{b}} + \Delta \dot{\mathcal{E}}^{\bar{a}\bar{b}}) (\dot{\mathcal{E}}_{\bar{a}\bar{b}} + \Delta \dot{\mathcal{E}}_{\bar{a}\bar{b}}). \quad (4.19)$$

Equations (4.17), (4.18), and (4.19) can be compared with the corresponding black-hole results displayed in Sec. I. We associate  $\dot{W}$  with  $\dot{M}$ ,  $\dot{J}$  with  $\dot{J}$ , and  $\dot{Q}$  with  $(\kappa/8\pi)\dot{A}$ , where  $\kappa$  is the black hole's surface gravity and  $A$  its surface area. With this association the energy-balance equation of Eq. (4.16) becomes  $\dot{M} = (\kappa/8\pi)\dot{A} + \Omega_H \dot{J}$ , the first law of black-hole mechanics. The details of this comparison were already described in Sec. I.

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